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1980 J. Phys. A: Math. Gen. 13 2465

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J. Phys. A: Math. Gen. 13 (1980) 2465–2477. Printed in Great Britain

Recursive method in one-dimensional Ising model

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Received 9 March 1979, in final form 2 January 1980

Abstract. In this paper we introduce a new recursive method which allows us to solve exactly one-dimensional Ising problems with higher-order interactions. The new method presents some important features, for example its exactness and the computational ease of its solution. We present a new method for evaluating the partition function. Moreover an application of this to the general approach is given and an analytic solution is found.

1. Introduction

As it is well known, the Ising model with nearest-neighbour interactions has attracted the attention of many investigations. In particular, in one dimension there are several methods for solving it, for example the matrix method and the generating function method. However, when more than the nearest-neighbour interactions are assumed or introduced in the physical problem, such methods turn out to be very complicated from the analytical point of view, since their solutions require approximations.

In this paper, we develop a new recursive method which allows us to obtain the exact analytical solution when higher-neighbour interactions are taken into account. Indeed, here we present as far as the third-neighbour interaction, but we indicate the method for possible higher interactions.

In § 4 we present a new method for evaluating the partition function in the thermodynamic limit. Finally, in § 5 an application of this method to the computation of the recursive matrices developed in § 3 is worked out. In this way an analytical solution of the partition function is found.

2. Recursive method for nearest-neighbour interaction

In this section we will introduce the recursive method for the usual Ising model with only nearest-neighbour interactions, extending it later.

As it is well known (Thompson 1972), let us consider the usual Ising model in one dimension. In a configuration $\{\mu\}$ the interaction energy is defined by

$$E\{\mu\} = -J \sum_{i=1}^{n-1} \mu_i\mu_{i+1} - H \sum_{i=1}^{n} \mu_i$$

where μ_i takes the value $+1$ or -1. μ_i denotes the spin value in the ith site, J is the coupling constant and H is the external magnetic field.

Thus, the partition function is expressed as

$$Z=\sum_{\{\mu\}}\exp(-\beta E\{\mu\})=\sum_{\{\mu\}}\exp\left(\beta J\sum_{i=1}^{n-1}\mu_i\mu_{i+1}+\beta H\sum_{i=1}^{n}\mu_i\right)$$

where $\beta=1/kT$. This last expression can be written in the form (see Thompson 1972)

$$Z=K\sum_{\{\mu\}}\prod_{i=1}^{n-1}(1+\omega_{\bar J}\mu_i\mu_{i+1})\prod_{i=1}^{n}(1+\omega_{\bar H}\mu_i)$$

where $\bar J=\beta J$, $\bar H=\beta H$, $\omega_{\bar J}=\tanh\bar J$, $\omega_{\bar H}=\tanh\bar H$, and $K=(\cosh\bar J)^{n-1}(\cosh\bar H)^n$.

The above expression for the partition function can be written again as

$$Z=K\sum_{\{\mu^2\}}\left[\sum_{\{\mu_1\}}(1+\omega_{\bar J}\mu_1\mu_2)(1+\omega_{\bar H}\mu_1)\right]\prod_{i=2}^{n-1}(1+\omega_{\bar J}\mu_i\mu_{i+1})\prod_{i=2}^{n}(1+\omega_{\bar H}\mu_i)$$

where the second sum is the sum over the states configurations $\mu_1=\pm1$. The first sum is taken over all the configurations $\{\mu^2\}=\{\mu_2,\mu_3,\ldots,\mu_n\}$. Performing the second sum in the above expression, we obtain with $\alpha_1=1$ and $\beta_1=\omega_{\bar J}\omega_{\bar H}$

$$\begin{aligned}Z&=2K\sum_{\{\mu^3\}}\sum_{\{\mu_2\}}(\alpha_1+\beta_1\mu_2)\prod_{i=2}^{n-1}(1+\omega_{\bar J}\mu_i\mu_{i+1})\prod_{i=2}^{n}(1+\omega_{\bar H}\mu_i)\\
&=2K\sum_{\{\mu^3\}}\left[\sum_{\{\mu_2\}}(\alpha_1+\beta_1\mu_2)(1+\omega_{\bar J}\mu_2\mu_3)(1+\omega_{\bar H}\mu_2)\right]\\
&\quad\times\prod_{i=3}^{n-1}(1+\omega_{\bar J}\mu_i\mu_{i+1})\sum_{i=3}^{n}(1+\omega_{\bar H}\mu_i)\\
&=2^2K\sum_{\{\mu^3\}}\left[(\alpha_1+\beta_1\omega_{\bar H})+(\beta_1\omega_{\bar J}+\alpha_1\omega_{\bar J}\omega_{\bar H})\mu_3\right]\\
&\quad\times\prod_{i=3}^{n-1}(1+\omega_{\bar J}\mu_i\mu_{i+1})\prod_{i=3}^{n}(1+\omega_{\bar H}\mu_i)\end{aligned}$$

where

$$\alpha_2=\alpha_1+\beta_1\omega_{\bar H}\qquad\text{and}\qquad\beta_2=\beta_1\omega_{\bar J}+\alpha_1\omega_{\bar J}\omega_{\bar H}.$$

Repeating this procedure it is easy to see that for μ_3, μ_4, as far as μ_{n-1}, we can obtain the recursion equations

$$\alpha_{i+1}=\alpha_i+\beta_i\omega_{\bar H},\qquad\beta_{i+1}=\beta_i\omega_{\bar J}+\alpha_i\omega_{\bar J}\omega_{\bar H}$$

since the mechanism is recursive. In addition we have for $j\leqslant n-1$

$$Z=2^jK\sum_{\{\mu^{j+1}\}}(\alpha_j+\beta_j\mu_{j+1})\prod_{i=j+1}^{n-1}(1+\omega_{\bar J}\mu_i\mu_{i+1})\prod_{i=j+1}^{n}(1+\omega_{\bar H}\mu_i).$$

As a particular case for $j=n-1$, we obtain

$$\begin{aligned}Z&=2^{n-1}K\sum_{\{\mu_n\}}(\alpha_{n-1}+\beta_{n-1}\mu_n)(1+\omega_{\bar H}\mu_n)\\
&=2^nK(\alpha_{n-1}+\beta_{n-1}\omega_{\bar H})=2^nK\alpha_n.\end{aligned}$$

Going back to the coefficients α and β, the formula (1) can be expressed in matrix form as

$$\begin{pmatrix}\alpha_{i+1}\\ \beta_{i+1}\end{pmatrix}=\begin{pmatrix}1&\omega_{\bar H}\\ \omega_{\bar J}\omega_{\bar H}&\omega_{\bar J}\end{pmatrix}\begin{pmatrix}\alpha_i\\ \beta_i\end{pmatrix}=A\begin{pmatrix}\alpha_i\\ \beta_i\end{pmatrix}.$$

This recursive coupled system can be uncoupled easily and solved accordingly. Indeed, consider the matrix C such that

$$C^{-1}AC = \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

where λ_1 and λ_2 are the eigenvalues of A and take the form

$$\lambda_{1,2} = \tfrac{1}{2}(1+\omega_{\bar{J}}) \pm \tfrac{1}{2}[(\omega_{\bar{J}}-1)^2 + 4\omega_{\bar{J}}\omega_{\bar{H}}^2]^{1/2}.$$

The discriminant $\Delta = (\omega_{\bar{J}}-1)^2 + 4\omega_{\bar{J}}\omega_{\bar{H}}^2$ is always positive under the condition $|\omega_{\bar{H}}| < 1$. This is always satisfied. Thus both eigenvalues are real and different.

A suitable C is

$$\cdot C = \begin{pmatrix} 1 & 1 \\ (1-\lambda_1)/\omega_{\bar{H}} & (1-\lambda_2)/\omega_{\bar{H}} \end{pmatrix}$$

whose columns are the corresponding eigenvectors. Therefore the system

$$\lambda_1 u_i = u_{i+1}, \qquad \lambda_2 v_i = v_{i+1},$$

whose solutions are given by

$$u_{i+1} = (\lambda_1)^i u_1 \qquad \text{and} \qquad v_{i+1} = (\lambda_2)^i v_1,$$

is related to the previous system of A by $\begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} = C\begin{pmatrix} u_i \\ v_i \end{pmatrix}$ or

$$\alpha_i = \lambda_1^i u_1 + \lambda_2^i v_1, \qquad \beta_i = \left(\frac{1-\lambda_1}{\omega_{\bar{H}}}\right) u_1 + \left(\frac{1-\lambda_2}{\omega_{\bar{J}}}\right) v_1$$

where

$$1 = \lambda_1 u_1 + \lambda_2 v_1, \qquad \omega_{\bar{J}}\omega_{\bar{H}} = \left(\frac{1-\lambda_1}{\omega_{\bar{H}}}\right) u_1 + \left(\frac{1-\lambda_2}{\omega_{\bar{J}}}\right) v_1.$$

Solving the latter equations for u_1 and v_1 and replacing them in the above equations, one can obtain the desired solution for α_i and β_i. With these values the partition function is completely determined.

3. Recursive method with two-neighbour interactions

Having shown the method for the nearest-neighbour interaction, in this section we shall explain how the recursive method can also be applied to two-neighbour interactions.

In this case the partition function takes the form

$$Z = K \sum_{\{\mu\}} \prod_{i=1}^{n} (1+\omega_{\bar{H}}\mu_i) \prod_{i=1}^{n-1} (1+\omega_{\bar{J}_1}\mu_i\mu_{i+1}) \prod_{i=1}^{n-2} (1+\omega_{\bar{J}_2}\mu_i\mu_{i+2})$$

where $K = (\cosh \bar{H})^n (\cosh \bar{J}_1)^{n-1} (\cosh \bar{J}_2)^{n-2}$, $\bar{J}_1 = J_1/kT$, $\bar{J}_2 = J_2/kT$ and $\bar{H} = H/kT$. Again H is the magnetic field applied to the system and J_1 and J_2 are the first- and second-neighbour coupling constants respectively.

Taking the partition function as before, we have

$$Z = K \sum_{\{\mu^2\}} \left[\sum_{\{\mu_1\}} (1+\omega_{\bar H}\mu_1)(1+\mu_{\bar J_1}\mu_1\mu_2)(1+\omega_{\bar J_2}\mu_1\mu_3)\right]$$

$$\times \prod_{i=2}^{n} (1+\omega_{\bar H}\mu_i) \prod_{i=2}^{n-1} (1+\omega_{\bar J_1}\mu_i\mu_{i+1}) \prod_{i=2}^{n-2} (1+\omega_{\bar J}\mu_i\mu_{i+1})$$

$$= 2K \sum_{\{\mu^2\}} (\alpha_1+\beta_1+\gamma_1\mu_3+\delta_1\mu_2\mu_3) \prod_{i=2}^{n} (1+\omega_{\bar H}\mu_i)$$

$$\times \prod_{i=2}^{n-1} (1+\omega_{\bar J_1}\mu_i\mu_{i+1}) \prod_{i=2}^{n-2} (1+\omega_{\bar J_2}\mu_i\mu_{i+2})$$

$$= 2K \sum_{\{\mu^3\}} \left[\sum_{\{\mu_2\}} (\alpha_1+\beta_1\mu_2+\gamma_1\mu_3+\delta_1\mu_2\mu_3)\right.$$

$$\left.\times (1+\omega_{\bar H}\mu_2)(1+\omega_{\bar J_1}\mu_2\mu_3)(1+\omega_{\bar J_2}\mu_2\mu_4)\right]$$

$$\times \prod_{i=3}^{n} (1+\omega_{\bar H}\mu_i) \prod_{i=3}^{n-1} (1+\omega_{\bar J_1}\mu_i\mu_{i+1}) \prod_{i=3}^{n-2} (1+\omega_{\bar J_2}\mu_i\mu_{i+2})$$

$$= 2^2 K \sum_{\{\mu^3\}} \left[(\alpha_1+\delta_1\omega_{J_1}+\gamma_1\omega_{\bar H}\omega_{\bar J_1}+\beta_1\omega_{\bar H})\right.$$

$$+ (\gamma_1+\delta_1\omega_{\bar H}+\beta_1\omega_{\bar J_1}+\alpha_1\omega_{\bar H}\omega_{\bar J_1})\mu_3$$

$$+ (\beta_1\omega_{\bar J_2}+\alpha_1\omega_{\bar H}\omega_{\bar J_2}+\gamma_1\omega_{\bar J_1}\omega_{\bar J_2}+\delta_1\omega_{\bar H}\omega_{\bar J_1}\omega_{\bar J_2})\mu_4$$

$$\left.+ (\gamma_1\omega_{\bar H}\omega_{\bar J_2}+\delta_1\omega_{\bar J_2}+\alpha_1\omega_{\bar J_1}\omega_{\bar J_2}+\beta_1\omega_{\bar H}\omega_{\bar J_1}\omega_{\bar J_2})\mu_3\mu_4\right]$$

$$\times \prod_{i=3}^{n} (1+\omega_{\bar H}\mu_i) \prod_{i=3}^{n-1} (1+\omega_{\bar J_1}\mu_i\mu_{i+1}) \prod_{i=3}^{n-2} (1+\omega_{\bar J_2}\mu_i\mu_{i+2})$$

where

$$\alpha_1 = 1, \qquad \beta_1 = \omega_{\bar H}\omega_{\bar J_1}, \qquad \gamma_1 = \omega_{\bar H}\omega_{\bar J_2} \qquad \text{and} \qquad \delta_1 = \omega_{\bar J_1}\omega_{\bar J_2}.$$

We denote by Z_3 the last term with the products in the last expression. Iterating the previous procedure we can easily obtain

$$Z = 2^i K \sum_{\{\mu^{i+1}\}} (\alpha_i+\beta_i\mu_{i+1}+\gamma_i\mu_{i+2}+\delta_i\mu_{i+1}\mu_{i+2}) Z_{i+1}$$

with $i+1 \leqslant n-2$, where the coefficients are recursively given by the matrix relation

$$\begin{pmatrix} \alpha_{i+1} \\ \beta_{i+1} \\ \gamma_{i+1} \\ \delta_{i+1} \end{pmatrix} = \begin{pmatrix} 1 & \omega_{\bar H} & \omega_{\bar H}\omega_{\bar J_1} & \omega_{\bar J_1} \\ \omega_{\bar H}\omega_{\bar J_1} & \omega_{\bar J_1} & 1 & \omega_{\bar H} \\ \omega_{\bar H}\omega_{\bar J_2} & \omega_{\bar J_2} & \omega_{\bar J_1}\omega_{\bar J_2} & \omega_{\bar H}\omega_{\bar J_1}\omega_{\bar J_2} \\ \omega_{\bar J_1}\omega_{J_2} & \omega_{\bar H}\omega_{\bar J_1}\omega_{\bar J_2} & \omega_{\bar H}\omega_{\bar J_2} & \omega_{\bar J_2} \end{pmatrix} \begin{pmatrix} \alpha_i \\ \beta_i \\ \gamma_i \\ \delta_i \end{pmatrix}.$$

To obtain the partition function effectively we take $i=n-2$, and therefore we perform the following operations:

$$Z=2^{n-2}K\sum_{\{\mu^{n-1}\}}(\alpha_{n-2}+\beta_{n-2}\mu_{n-1}+\gamma_{n-2}\mu_n+\delta_{n-2}\mu_{n-1}\mu_n)$$
$$\times(1+\omega_{\bar{H}}\mu_{n-1})(1+\omega_{\bar{H}}\mu_n)(1+\omega_{\bar{J}_1}\mu_{n-1}\mu_n)$$
$$=2^{n-1}K\sum_{\{\mu^n\}}[(\alpha_{n-1}+\beta_{n-1}\mu_n)(1+\omega_{\bar{H}}\mu_n)]=2^nK(\alpha_{n-1}+\omega_{\bar{H}}\beta_{n-1}).$$

In this way we have obtained in a recursive way the final expression.

In order to compute effectively the coefficients in the last expression of the partition function it is necessary to solve the recurrence system given above.

However, if it is not necessary to have an explicit and analytical expression of the entries of the recurrent system, then the aim of having the partition function, at this point, becomes very simple. We only need to have a suitable numerical iteration which can be performed accordingly. We remark that when an analytical solution is necessary this might be done by uncoupling the recurrence system, similarly as was performed in the previous paragraph. But here the difficulties will increase enormously since it turns out that the roots of an equation of degree four must be calculated. This is a great computational task, which we will not study due to its difficulties. However, we will present a recursive procedure which allows us to obtain an analytical exact solution.

Let us denote by I the set of entries (1, 2), (1, 3), (2, 1), (2, 4), (3, 1), (3, 4), (4, 2) and (4, 3); then the matrix A given above for the recursive expression of α_i, β_i, γ_i, δ_i, has $\omega_{\bar{H}}$ as a factor for $(k, l)\in\mathrm{I}$ and no factor $\omega_{\bar{H}}$ for $(k, l)\notin\mathrm{I}$. Thus, it is possible to see that the even power A^{2p} has entries

$$\sum_{m=0}^{p}a_{kl}(2p,m)\omega_{\bar{H}}^{2m},\qquad \sum_{m=0}^{p-1}a_{kl}(2p,m)\,\omega_{\bar{H}}^{2m+1},$$

for $(k, l)\notin\mathrm{I}$ and $(k, l)\in\mathrm{I}$, respectively.

Similarly, for the odd power A^{2p+1}, this matrix has entries

$$\sum_{m=0}^{p}a_{kl}(2p+1,m)\omega_{\bar{H}}^{2m},\qquad \sum_{m=0}^{p}a_{kl}(2p+1,m)\omega_{\bar{H}}^{2m+1},$$

for $(k, l)\notin\mathrm{I}$ and $(k, l)\in\mathrm{I}$ respectively.

At this point, we point out that the importance of having the power matrices explicitly as polynomials in $\omega_{\bar{H}}$ is due to the fact that in applied problems related to properties of thermodynamic systems there appears the necessity of studying the variation of certain entries with respect to $\bar{H}$.

The coefficients of the polynomials expressing the entries of the power matrices can be computed recursively as follows, with $A=(a_{kl})$:

$$a_{kl}(2p+1,0)=\sum_{n=1,4}a_{kn}(2p,0)a_{nl},$$

$$a_{kl}(2p+1,m)=\sum_{n=1,4}a_{kn}(2p,m)a_{nl}+\sum_{n=2,3}a_{kn}(2p,m-1)a_{nl},\qquad m\geqslant 1,$$

for $(k, l) \notin \mathrm{I}$ and $p \geq 1$. For the remaining entries we have

$$a_{kl}(2p+1, m) = \sum_{n=1,4} a_{kn}(2p, m)a_{nl} + \sum_{n=2,3} a_{kn}(2p, m)a_{nl}, \qquad m \neq p,$$

$$a_{kl}(2p+1, p) = \sum_{n=1,4} a_{kn}(2p, p)a_{nl},$$

for any $(k, l) \in \mathrm{I}$ and $p \geq 1$.

Furthermore, for even powers the entries are computed as

$$a_{kl}(2p+2, 0) = \sum_{n=1,4} a_{kn}(2p+1, 0)a_{nl},$$

$$a_{kl}(2p+2, m) = \sum_{n=1,4} a_{kn}(2p+1, m)a_{nl} + \sum_{n=2,3} a_{kn}(2p+1, m-1)a_{nl}, \qquad 1 \leq m \leq p,$$

$$a_{kl}(2p+2, p+1) = \sum_{n=2,3} a_{kn}(2p+1, p)a_{nl},$$

for $(k, l) \notin \mathrm{I}$ and finally

$$a_{kl}(2p+2, m) = \sum_{n=1}^{4} a_{kl}(2p+1, m)a_{nl}$$

for $(k, l) \in \mathrm{I}$.

4. Recursive method with three-neighbour interactions

In this section we will extend the model to when a third interaction is included in the physical model. In this case, we have that the partition function is reduced to the form

$$Z = K \sum_{\{\mu\}} \prod_{i=1}^{n} (1 + \omega_{\bar{H}}\mu_i) \prod_{i=1}^{n-1} (1 + \omega_{\bar{J}_1}\mu_i\mu_{i+1}) \prod_{i=1}^{n-2} (1 + \omega_{\bar{J}_2}\mu_i\mu_{i+2}) \prod_{i=1}^{n-3} (1 + \omega_{\bar{J}_3}\mu_i\mu_{i+3})$$

$$= K \sum_{\{\mu\}} Z_1\{\mu^1\}$$

where $\bar{J}_3$ is the third-neighbour coupling constant and in K there appears a further term $(\cosh \bar{J}_3)^{n-3}$.

Similarly, as in the previous cases, we apply the general idea of recursion taking in the first step all the terms with μ_1:

$$Z = K \sum_{\{\mu^2\}} \sum_{\{\mu_1\}} (1 + \omega_{\bar{H}}\mu_1)(1 + \omega_{\bar{J}_1}\mu_1\mu_2)(1 + \omega_{\bar{J}_2}\mu_1\mu_3)(1 + \omega_{\bar{J}_3}\mu_1\mu_4)Z_2\{\mu^2\}$$

$$= 2K \sum_{\{\mu^2\}} (\theta_1^{(0)} + \theta_1^{(1)}\mu_2 + \theta_1^{(2)}\mu_3 + \theta_1^{(3)}\mu_4 + \theta_1^{(4)}\mu_2\mu_3 + \theta_1^{(5)}\mu_2\mu_4 + \theta_1^{(6)}\mu_3\mu_4 + \theta_1^{(7)}\mu_2\mu_3\mu_4)Z_2\{\mu^2\}$$

where

$\theta_1^{(0)} = 1, \qquad \theta_1^{(1)} = \omega_{\bar{H}}\omega_{\bar{J}_1}, \qquad \theta_1^{(2)} = \omega_{\bar{H}}\omega_{\bar{J}_2}, \qquad \theta_1^{(3)} = \omega_{\bar{H}}\omega_{\bar{J}_3},$

$\theta_1^{(4)} = \omega_{\bar{J}_1}\omega_{\bar{J}_2}, \qquad \theta_1^{(5)} = \omega_{\bar{J}_1}\omega_{\bar{J}_3}, \qquad \theta_1^{(6)} = \omega_{\bar{J}_2}\omega_{\bar{J}_3}, \qquad \theta_1^{(7)} = \omega_{\bar{H}}\omega_{\bar{J}_1}\omega_{\bar{J}_2}\omega_{\bar{J}_3}.$

Continuing with the recursive method, one can obtain inductively the following expressions:

$$
\begin{aligned}
Z = 2^i K \sum_{\{\mu^{i+1}\}} & (\theta_i^{(0)} + \theta_i^{(1)}\mu_{i+1} + \theta_i^{(2)}\mu_{i+2} + \theta_i^{(3)}\mu_{i+3} + \theta_i^{(4)}\mu_{i+1}\mu_{i+2} \\
& + \theta_i^{(5)}\mu_{i+1}\mu_{i+3} + \theta_i^{(6)}\mu_{i+2}\mu_{i+3} + \theta_i^{(7)}\mu_{i+1}\mu_{i+2}\mu_{i+3}) Z_{i+1}\{\mu^{i+1}\} \\
= 2^i K \sum_{\{\mu^{i+2}\}} \sum_{\{\mu_{i+1}\}} & [(\theta_i^{(0)} + \theta_i^{(1)}\mu_{i+1} + \theta_i^{(2)}\mu_{i+2} + \theta_i^{(3)}\mu_{i+3} + \theta_i^{(4)}\mu_{i+1}\mu_{i+2} \\
& + \theta_i^{(5)}\mu_{i+1}\mu_{i+3} + \theta_i^{(6)}\mu_{i+2}\mu_{i+3} + \theta_i^{(7)}\mu_{i+1}\mu_{i+2}\mu_{i+3})(1 + \omega_{\bar H}\mu_{i+1}) \\
& \times (1 + \omega_{\bar J_1}\mu_{i+1}\mu_{i+2})(1 + \omega_{\bar J_2}\mu_{i+1}\mu_{i+3})(1 + \omega_{\bar J_3}\mu_{i+1}\mu_{i+4})] Z_{i+2}\{\mu^{i+2}\} \\
= 2^{i+1} K \sum_{\{\mu^{i+2}\}} & (\theta_{i+1}^{(0)} + \theta_{i+1}^{(1)}\mu_{i+2} + \theta_{i+1}^{(2)}\mu_{i+3} + \theta_{i+1}^{(3)}\mu_{i+4} + \theta_{i+1}^{(4)}\mu_{i+2}\mu_{i+3} \\
& + \theta_{i+1}^{(5)}\mu_{i+2}\mu_{i+4} + \theta_{i+1}^{(6)}\mu_{i+3}\mu_{i+4} + \theta_{i+1}^{(7)}\mu_{i+2}\mu_{i+3}\mu_{i+4}) Z_{i+2}\{\mu^{i+2}\}
\end{aligned}
$$

with $i \leqslant n-3$, where the new θ's are obtained from the previous ones by a matrix multiplication, namely:

$$
\begin{pmatrix} \theta_{i+1}^{(0)} \\ \theta_{i+1}^{(1)} \\ \theta_{i+1}^{(2)} \\ \theta_{i+1}^{(3)} \\ \theta_{i+1}^{(4)} \\ \theta_{i+1}^{(5)} \\ \theta_{i+1}^{(6)} \\ \theta_{i+1}^{(7)} \end{pmatrix}
=
\begin{pmatrix}
1 & \omega_{\bar H} & \omega_{\bar H}\omega_{\bar J_1} & \omega_{\bar H}\omega_{\bar J_2} & \omega_{\bar J_1} & \omega_{\bar J_2} & \omega_{\bar J_1}\omega_{\bar J_2} & \omega_{\bar H}\omega_{\bar J_1}\omega_{\bar J_2} \\
\omega_{\bar H}\omega_{\bar J_1} & \omega_{\bar J_1} & 1 & \omega_{\bar J_1}\omega_{\bar J_2} & \omega_{\bar H} & \omega_{\bar H}\omega_{\bar J_1}\omega_{\bar J_2} & \omega_{\bar H}\omega_{\bar J_2} & \omega_{\bar J_2} \\
\omega_{\bar H}\omega_{\bar J_2} & \omega_{\bar J_2} & \omega_{\bar J_1}\omega_{\bar J_2} & 1 & \omega_{\bar H}\omega_{\bar J_1}\omega_{\bar J_2} & \omega_{\bar H} & \omega_{\bar H}\omega_{\bar J_1} & \omega_{\bar J_1} \\
\omega_{\bar H}\omega_{\bar J_3} & \omega_{\bar J_3} & \omega_{\bar J_1}\omega_{\bar J_3} & \omega_{\bar J_2}\omega_{\bar J_3} & \omega_{\bar H}\omega_{\bar J_1}\omega_{\bar J_3} & \omega_{\bar H}\omega_{\bar J_2}\omega_{\bar J_3} & \omega_{\bar H}\omega_{\bar J_1}\omega_{\bar J_2}\omega_{\bar J_3} & \omega_{\bar J_1}\omega_{\bar J_2}\omega_{\bar J_3} \\
\omega_{\bar J_1}\omega_{\bar J_2} & \omega_{\bar H}\omega_{\bar J_1}\omega_{\bar J_2} & \omega_{\bar H}\omega_{\bar J_2} & \omega_{\bar H}\omega_{\bar J_1} & \omega_{\bar J_2} & \omega_{\bar J_1} & 1 & \omega_{\bar H} \\
\omega_{\bar J_1}\omega_{\bar J_3} & \omega_{\bar H}\omega_{\bar J_1}\omega_{\bar J_3} & \omega_{\bar H}\omega_{\bar J_3} & \omega_{\bar H}\omega_{\bar J_1}\omega_{\bar J_2}\omega_{\bar J_3} & \omega_{\bar J_3} & \omega_{\bar J_1}\omega_{\bar J_2}\omega_{\bar J_3} & \omega_{\bar J_2}\omega_{\bar J_3} & \omega_{\bar H}\omega_{\bar J_2}\omega_{\bar J_3} \\
\omega_{\bar J_2}\omega_{\bar J_3} & \omega_{\bar H}\omega_{\bar J_2}\omega_{\bar J_3} & \omega_{\bar H}\omega_{\bar J_1}\omega_{\bar J_2}\omega_{\bar J_3} & \omega_{\bar H}\omega_{\bar J_3} & \omega_{\bar J_1}\omega_{\bar J_2}\omega_{\bar J_3} & \omega_{\bar J_3} & \omega_{\bar J_1}\omega_{\bar J_3} & \omega_{\bar H}\omega_{\bar J_1}\omega_{\bar J_3} \\
\omega_{\bar H}\omega_{\bar J_1}\omega_{\bar J_2}\omega_{\bar J_3} & \omega_{\bar J_1}\omega_{\bar J_2}\omega_{\bar J_3} & \omega_{\bar J_2}\omega_{\bar J_3} & \omega_{\bar J_1}\omega_{\bar J_3} & \omega_{\bar H}\omega_{\bar J_2}\omega_{\bar J_3} & \omega_{\bar H}\omega_{\bar J_1}\omega_{\bar J_3} & \omega_{\bar H}\omega_{\bar J_3} & \omega_{\bar J_3}
\end{pmatrix}
\begin{pmatrix} \theta_i^{(0)} \\ \theta_i^{(1)} \\ \theta_i^{(2)} \\ \theta_i^{(3)} \\ \theta_i^{(4)} \\ \theta_i^{(5)} \\ \theta_i^{(6)} \\ \theta_i^{(7)} \end{pmatrix}.
$$

In order to obtain the complete partition function we have to compute the 'queue' of the recursive expression which contributes the end effect.

$$
\begin{aligned}
Z = 2^{n-3} K \sum_{\{\mu^{n-2}\}} & (\theta_{n-3}^{(0)} + \theta_{n-3}^{(1)}\mu_{n-2} + \theta_{n-3}^{(2)}\mu_{n-1} + \theta_{n-3}^{(3)}\mu_n + \theta_{n-3}^{(4)}\mu_{n-2}\mu_{n-1} \\
& + \theta_{n-3}^{(5)}\mu_{n-2}\mu_n + \theta_{n-3}^{(6)}\mu_{n-1}\mu_n + \theta_{n-3}^{(7)}\mu_{n-2}\mu_{n-1}\mu_n) \\
& \times [(1 + \omega_{\bar H}\mu_{n-2})(1 + \omega_{\bar J_1}\mu_{n-2}\mu_{n-1})(1 + \omega_{\bar J_2}\mu_{n-2}\mu_n)] \\
& \times [(1 + \omega_{\bar H}\mu_{n-1})(1 + \omega_{\bar J_1}\mu_{n-1}\mu_n)](1 + \omega_{\bar H}\mu_n)
\end{aligned}
$$

$$
\begin{aligned}
&= 2^{n-2}K \sum_{\{\mu^{n-1}\}} (\theta^{(0)}_{n-2} + \theta^{(1)}_{n-2}\mu_{n-1} + \theta^{(2)}_{n-2}\mu_n + \theta^{(4)}_{n-2}\mu_{n-1}\mu_n) \\
&\quad \times [(1+\omega_{\bar{H}}\mu_{n-1})(1+\omega_{\bar{J}_1}\mu_{n-1}\mu_n)](1+\omega_{\bar{H}}\mu_n) \\
&= 2^{n-1}K \sum_{\{\mu^n\}} [(\theta^{(0)}_{n-2} + \theta^{(1)}_{n-2}\omega_{\bar{H}} + \theta^{(2)}_{n-2}\omega_{\bar{H}}\omega_{\bar{J}_1} + \theta^{(4)}_{n-2}\omega_{\bar{J}_1}) \\
&\quad + (\theta^{(0)}_{n-2}\omega_{\bar{H}}\omega_{\bar{J}_1} + \theta^{(1)}_{n-2}\omega_{\bar{J}_1} + \theta^{(2)}_{n-2} + \theta^{(4)}_{n-2}\omega_{\bar{H}})\mu_n](1+\omega_{\bar{H}}\mu_n) \\
&= 2^n K[(\theta^{(0)}_{n-2} + \theta^{(1)}_{n-2}\omega_{\bar{H}} + \theta^{(2)}_{n-2}\omega_{\bar{H}}\omega_{\bar{J}_1} + \theta^{(4)}_{n-2}\omega_{\bar{J}_1}) \\
&\quad + (\theta^{(0)}_{n-2}\omega_{\bar{H}}\omega_{\bar{J}_1} + \theta^{(1)}_{n-2}\omega_{\bar{J}_1} + \theta^{(2)}_{n-2} + \theta^{(4)}_{n-2})\omega_{\bar{H}}].
\end{aligned}
$$

The first equality is obtained by summing over μ^{n-2}, the next over μ^{n-1} and finally over μ^n.

From here, as in the previous case, if one only desires numerical evaluation for the partition function, the θ's can be obtained directly by numerical iteration of the recurrence matrix to the power $n-2$ and performing the necessary operation as shown in the last expression of Z.

In the case when an analytical expression for the entries of the recurrent matrix is required, say in terms of $\omega_{\bar{H}}$, a similar analysis as in the previous case of two-neighbour interactions might easily be performed. Indeed, the structure is kept the same with the only variation being in the polynomial formation. Here also appears a block distribution as in the previous case.

The recurrence relations for the polynomial coefficients have an analogous form as before.

5. A new general method

The method presented here represents an alternative way for evaluating the partition function in a very general case. As is well known, when the size of the matrix is greater than 2×2, obtaining the maximum eigenvalue is not always possible. Our method has the advantage of finding any one of the entries of a matrix (which in the thermodynamic limit is equivalent to the fact of knowing the maximum eigenvalue) as related to a submatrix of inferior size.

Now we will introduce our method. Consider a matrix a_{kl}, $k, l: 1, \ldots, N$ and let $a_{kl}(j)$ be the (k, l)th entry in the matrix A^j. Therefore, by definition of the matrix product, we have

$$
a_{kl}(j) = \sum_{p_1=1}^{N} a_{kp_1}a_{p_1l}(j-1) = a_{kk}a_{kl}(j-1) + \sum_{\substack{p_1=1 \\ p_1\neq k}}^{N} a_{kp_1}a_{p_1l}(j-1)
$$

and in general for some other p_1 we have

$$
a_{p_1l}(j) = \sum_{p_2=1}^{N} a_{p_1p_2}a_{p_2l}(j-1) = a_{p_1k}a_{kl}(j-1) + \sum_{\substack{p_2=1 \\ \neq k}}^{N} a_{p_1p_2}a_{p_2l}(j-1).
$$

Now replacing the value $a_{p_1l}(j-1)$ given by the latter equation in the previous one, we obtain

$$
a_{kl}(j) = a_{kk}a_{kl}(j-1) + \sum_{\substack{p_1=1 \\ \neq k}}^{N} a_{kp_1}a_{p_1k}a_{kl}(j-2) + \sum_{p_1\neq k}^{N}\sum_{p_2\neq k}^{N} a_{kp_1}a_{p_1p_2}a_{p_2l}(j-2)
$$

and by recursion this gives in general

$$a_{kl}(j)=a_{kk}a_{kl}(j-1)+a_k a^k a_{kl}(j-2)+a_k\bar{A}a^k a_{kl}(j-3)$$
$$+\ldots+a_k\bar{A}^{j-3}a^k a_{kl}(1)+a_k\bar{A}^{j-2}a^l \qquad (1)$$

where the matrix $\bar{A}$ is obtained from A by deleting the row k and the column k. The vector a_k is the kth row of A with the element k deleted. Similarly a_m is the mth column of A without the entry k, for $m=k, l$. The products are matrix products and scalar products.

The latter expression relates the elements of the power matrix $a_{kl}(j)$ with the elements of the power matrices $\bar{A}^m$. At this point it is important to mention that the powers of $\bar{A}$ are all less than j and its size is one less than A.

The recursion in the expression (1) is of the type

$$n_k=a^k_{k-1}n_{k-1}+a^k_{k-2}n_{k-2}+\ldots+a^k_0 n_0+b_k$$

where the unknowns are the n's and the a's are all known coefficients. This general problem was solved recently (Marchi and Millan 1977), and the solution may be written analytically as

$$n_k=\left(\sum_{r=1}^{k}\sum_{(l_1,\ldots,l_r)^k\in E_r^k}a^k_{(l_1,\ldots,l_r)}\right)n_0+\left[\sum_{i=1}^{k-1}\left(\sum_{r=1}^{k-i}\sum_{(l_1,\ldots,l_r)^{k-i}\in E_r^k}a^k_{(l_1,\ldots,l_r)^{k-i}}\right)b_i\right]+b_k \qquad (2)$$

where the sets are given by

$$E_r^k=\left\{(l_1,\ldots,l_r):\sum_{i=1}^{r}l_i=k,\ l_i>0\text{ and integers}\right\}$$

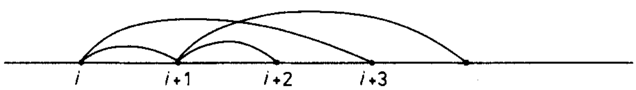
and the coefficients by

$$a^k_{(l_1,\ldots,l_r)^k}=a^k_{k-l_1}a^{k-l_1}_{k-l_1-l_2}\ldots a^{k-(l_1+l_2+\ldots+l_{r-1})}_{k-k}.$$

6. An application

In § 3 we have obtained the partition function as a function of the recursive coefficients θ. These coefficients are obtained from the iteration of a matrix, recursively.

We will now study a particular case of that matrix when $\omega_H=\omega_{J_2}=0$. This case is represented schematically by



where the contribution to the partition function is given by interaction of the sites i, $i+1$; i, $i+3$ etc.

We will solve this particular case in an exact and analytical way using the method presented in the previous section.

It follows from the fact that $\omega_H=\omega_{J_2}=0$ that the general matrix (written as in § 3) reduces to the study of the maximum eigenvalue of the following matrix:

$$A=\begin{pmatrix}1 & \omega_{J_1} & 0 & 0\\ 0 & 0 & \omega_{J_1} & 1\\ \omega_{J_3}\omega_{J_1} & \omega_{J_3} & 0 & 0\\ 0 & 0 & \omega_{J_3} & \omega_{J_3}\omega_{J_1}\end{pmatrix}.$$

But we will now prove that the maximum eigenvalue in the thermodynamic limit is equivalent to the fact of knowing any of the entries of the matrix.

Consider the partition function to be given by the expression

$$Z=\sum_{j=1}^{k}\sum_{l=1}^{k} b_{1j}a_{jl}(N)c_{l1};$$

then by taking the natural logarithm,

$$\frac{1}{N}\ln Z=\frac{1}{N}\ln a_{kl}(N)+\frac{1}{N}\ln\left(\sum_{j=1}^{k}\sum_{l=1}^{k}\frac{b_{1j}a_{jl}(N)c_{l1}}{a_{kl}(N)}\right)$$

with

$$a_{kl}(N)=\sum_{r=1}^{k} c_{kl}^{r}\lambda_r^N$$

where λ_r are the eigenvalues of the matrix A and the c's are coefficients. Arranging the eigenvalues in the manner $\lambda_1\geqslant\lambda_2\geqslant\ldots\geqslant\lambda_k$ it is easy to see that

$$\lim_{N\to\infty}\frac{1}{N}\ln Z=\lim_{N\to\infty}\frac{1}{N}\ln a_{kl}(N)=\lim_{N\to\infty}\ln\lambda_{\max}$$

because

$$(\lambda_i/\lambda_{\max})^N\to 0 \text{ or } 1 \qquad \text{if } \lambda_{\max}>\lambda_i \quad \text{or} \quad \lambda_{\max}=\lambda_i.$$

We now go back to the problem of solving any entry of the matrix given above with the method developed in the previous section.

The expression in (1) can be rewritten as

$$a_{kl}(j)=\sum_{i=1}^{j-1} c_i^j a_{kl}(i)+\beta_{kl}(j)$$

where the coefficients c are now given by

$$c_{j-l}^{j}=a_k\bar{A}^{l-2}a^k \qquad \text{for } l\geqslant 2,$$

$$c_{j-1}^{j}=a_{kk}.$$

As we have proved, it is the same to compute any entry in the thermodynamic limit, therefore we will study the entry (1,1) of the matrix A. Thus

$$a_{11}(j)=\sum_{i=1}^{j-1} c_i^j a_{11}(i)+\beta_{11}(j)$$

where now

$$c_{j-l}^{j}=\beta\bar{A}^{l-2}\gamma, \qquad l\geqslant 2,$$

$$c_{j-1}^{j}=a_{11}=1$$

and

$$\beta=(\omega_{J_1},0,0), \qquad \gamma=\begin{pmatrix}0\\ \omega_{J_3}\omega_{J_1}\\ 0\end{pmatrix}, \qquad \bar{A}=\begin{pmatrix}0 & \omega_{J_1} & 1\\ \omega_{J_3} & 0 & 0\\ 0 & \omega_{J_3} & \omega_{J_3}\omega_{J_1}\end{pmatrix}$$

which gives for any step

$$\beta \bar{A}^i \gamma = \omega_{J_3}\omega_{J_1}^2 \bar{a}_{12}(i), \qquad \beta_{11}(i) = \omega_{J_3}\omega_{J_1}^2 \bar{a}_{12}(i-2)$$

where $\bar{a}_{12}$ is the entry in the position (1, 2) of the matrix $\bar{A}$.

We remark that in this case the general formula (2) becomes

$$a_{11}(j) = \sum_{r=1}^{j-1} \sum_{(l_1,\dots,l_r)^{j-1}\in E_r^{j-1}} a^j_{(l_1,\dots,l_r)^{j-1}} a_{11}(1) + \left[\sum_{i=2}^{j-1}\left(\sum_{r=1}^{j-i} \sum_{(l_1,\dots,l_r)^{j-i}\in E_r^{j-i}} a^j_{(l_1,\dots,l_r)^{j-i}}\right)\beta_{11}(i)\right] + \beta_{11}(j). \tag{3}$$

We recall that

$$E_r^{j-1} = \left\{(l_1, \dots, l_r) : \sum_{i=1}^{r} l_i = j-1,\ l_i > 0 \text{ and integers}\right\}.$$

Next, our purpose is to compute explicitly the coefficients

$$a^j_{(l_1,\dots,l_r)^{\bar{j}}} = c^j_{j-l_1} c^{j-l_1}_{j-l_1-l_2} \dots c^{j-(l_1+l_2+\dots+l_{r-1})}_{j-\bar{j}}.$$

We show that if $r=1$, then $l_1 = j-1$ and

$$a^j_{(j-1)^{j-1}} = c^j_1 = \omega_{J_3}\omega_{J_1}^2 \bar{a}_{12}(j-3).$$

For $r=2$, we have $l_1+l_2 = j-1$ and only two cases arise with $l=1$, namely, $l_1=1$ and $l_2 = j-2$ or $l_1 = j-2$ and $l_2 = 1$. In both cases the result is the same:

$$a^j_{(l_1,l_2)^{j-1}} = \omega_{J_3}\omega_{J_1}^2 \bar{a}_{12}(j-4).$$

When either $l_1 = 2$ or $l_2 = 2$ the coefficient $a^j_{(l_1,l_2)^{j-1}} = 0$ because $\beta\gamma = 0$. In the remaining cases the coefficient under consideration turns out to be

$$a^j_{(l_1,l_2)^{j-1}} = \omega_{J_3}^2\omega_{J_1}^4 \bar{a}_{12}(l_1-2)\bar{a}_{12}(l_2-2).$$

In a similar way for $r=3$, it is easy to see that four cases arise.

(i) All the l_i are one. In this case we have $a^j_{(1,1,1)^3} = 1$.

(ii) Among the three l_i there are exactly two equal to one. In such a case the coefficient is the same for all the different cases:

$$a^j_{(l_1,l_2,l_3)^{j-1}} = \omega_{J_3}\omega_{J_1}^2 \bar{a}_{12}(j-5).$$

(iii) Among the three l_i there is only one equal to one. Therefore, in this case, we have

$$a^j_{(l_1,l_2,l_3)^{j-1}} = \omega_{J_3}^2\omega_{J_1}^4 \bar{a}_{12}(l_i-2)\bar{a}_{12}(l_{\bar{i}}-2),$$

where here l_i and $l_{\bar{i}}$ are those different to one. Again in this case (when an $l_i = 2$) the coefficient becomes zero.

(iv) All the l_i are different from one. Again if an l_i equals two the coefficient is zero. Otherwise

$$a^j_{(l_1,l_2,l_3)^{j-1}} = \omega_{J_3}^3\omega_{J_1}^6 \bar{a}_{12}(l_i-2)\bar{a}_{12}(l_2-2)\bar{a}_{12}(l_3-2).$$

An analogous analysis for greater r may show that similar cases of what we have already obtained will appear. Moreover, other terms will appear too. As an example, we show for $r=4$ that when all l_i are different from 1 and 2 the coefficient now is

$$a^j_{(l_1,l_2,l_3,l_4)^{j-1}} = \omega_{J_3}^4\omega_{J_1}^8 \bar{a}_{12}(l_1-2)\bar{a}_{12}(l_2-2)\bar{a}_{12}(l_3-2)\bar{a}_{12}(l_4-2).$$

From here it is easy to see that we may arrange all the coefficients having only one $\bar{a}_{12}$ in a sum which is

$$\omega_{J_3}\omega_{J_1}^2 \sum_{k=1}^{j-3} \binom{k}{1} \bar{a}_{12}[j-(k+2)]$$

where $\binom{k}{1}$ is the respective combinatorial number which appears as the number of different forms to combine $l_i = 1$. Now arranging all the coefficients with two $\bar{a}_{12}$, this sum is

$$\sum_{k=2}^{j-5} \omega_{J_3}^2 \omega_{J_1}^4 \sum_{l=3}^{j-(k+2)} \binom{k}{2} \bar{a}_{12}(l-2)\bar{a}_{12}[j-(l+k+1)].$$

Thus, following this procedure, it is possible to show that the final result is

$$\begin{aligned} a_{11}(j) = \Bigg(1 + & \sum_{p=1}^{[(j-1)/3]} \sum_{k=p}^{j-(2p+1)} \binom{k}{p} \omega_{J_3}^p \omega_{J_1}^{2p} \sum_{l_1=3}^{j-(k+(2p-2))} \sum_{l_2=3}^{j-(l_1+k+(2p-5))} \cdots \\ & \times \sum_{l_{p-1}=3}^{j-(l_1+l_2+\ldots+l_{p-2}+k+2p-3(p-1)+1)} \bar{a}_{12}(l_1-2)\bar{a}_{12}(l_2-2)\ldots \\ & \times \bar{a}_{12}\{j-(l_1+l_2+\ldots+l_{p-1}+k+2p-3(p-1))\}\Bigg) \\ & + \Bigg[\sum_{i=3}^{j-1}\Bigg(1+\sum_{p=1}^{[(j-i)/3]} \sum_{k=p}^{j-(2p+1)} \binom{k}{p} \omega_{J_3}^p \omega_{J_1}^{2p} \sum_{l_1=3}^{j-(k+(2p-2))} \cdots \\ & \times \sum_{l_{p-1}=3}^{j-(l_1+l_2+\ldots+l_{p-2}+k+2p-3(p-1)+1)} \bar{a}_{12}(l_1-2)\bar{a}_{12}(l_2-2)\ldots \\ & \times \bar{a}_{12}\{j-(l_1+l_2+\ldots+l_{p-1}+k+2p-3(p-1))\}\Bigg)\omega_{J_3}\omega_{J_1}^2\bar{a}_{12}(i-2)\Bigg] \\ & + \omega_{J_3}\omega_{J_1}^2\bar{a}_{12}(j-2), \end{aligned} \tag{4}$$

where $[x]$ stands for the largest integer $\leqslant x$.

Now, in order to finish our analysis to obtain the explicit solution, either one repeats the analysis for $\bar{a}_{12}$ with the same method using a 2×2 matrix or one directly computes the values of $\bar{a}_{12}$ by diagonalising the submatrix $\bar{A}$ to obtain

$$\bar{a}_{12}(j) = \sum_{r=1}^{3} c_r \lambda_r^j. \tag{5}$$

We choose the last approach. Consider the matrix

$$\bar{A} = \begin{pmatrix} 0 & \omega_{J_1} & 1 \\ \omega_{J_3} & 0 & 0 \\ 0 & \omega_{J_3} & \omega_{J_3}\omega_{J_1} \end{pmatrix};$$

its characteristic equation is

$$-\lambda^3 + a_1\lambda^2 + a_1\lambda + a_3 = 0$$

where

$$a_1 = \omega_{J_3}\omega_{J_1} \qquad \text{and} \qquad a_3 = (\omega_{J_3}^2 - a_1^2).$$

As usual, for the cubic equation, it can be changed to

$$\lambda^3 + p\lambda + q = 0$$

where

$$p=-(a_1+a_1^2/3), \qquad q=-(a_3+a_1^2/3+2a_1^3/27).$$

In order to know what type of solutions we have, it is important to study the discriminant

$$\Delta=p^3/27+q^2/4=-\alpha a_1^3+2\alpha a_1^4-\alpha a_1^5-\tfrac{1}{3}a_1^2\omega_{J_3}^2+\alpha\omega_{J_3}^2a_1^3+\omega_{J_3}^4/4$$

where

$$\alpha=1/27.$$

For simplicity we assume $\omega_{J_3}=\omega_{J_1}$, which implies that the interactions of the first and third neighbours are the same. It results that $\Delta>0$ for any $0<\omega_{J_1}^2<0{\cdot}8$. Therefore, we are faced with one real root and two complex conjugate eigenvalues

$$\lambda_1=u_1+v_1, \qquad \lambda_2=u_1\epsilon+v_1\bar{\epsilon}, \qquad \lambda_3=u_1\bar{\epsilon}+v_1\epsilon$$

where

$$u_1^3=[-q/2+(q^2/4+p^3/27)^{1/2}], \qquad v_1^3=[-q/2-(q^2/4+p^3/27)^{1/2}]$$

and

$$\epsilon=\tfrac{1}{2}(-1+\mathrm{i}\sqrt{3}), \qquad \bar{\epsilon}=\tfrac{1}{2}(-1-\mathrm{i}\sqrt{3}).$$

Knowing the eigenvalues, it remains to compute the coefficients in (5). It is easy to see that these are given by

$$c_r=\frac{\lambda_r^3(\lambda_r-\omega_{J_3}\omega_{J_1})^2}{2\omega_{J_3}\lambda_r^2(\lambda_r-\omega_{J_3}\omega_{J_1})^2+\omega_{J_3}^3\lambda_r}.$$

With all this we have found out an analytic solution of $\bar{a}_{12}(j)$ and, substituting this in (4), an analytic solution for the partition function in the thermodynamic limit was derived.

7. Final remarks

We would like to point out again that the recursive method presented here gives rise to an exact solution of the partition function or some of the functions derived from it. Moreover the method permits us to work with open chains. The exactness and effectiveness of it gives power to the method.

From a physical point of view the recursive method allows us to solve exactly many problems related to Ising models in one dimension.

Even though the computational approach of the new method of solution introduced in this paper is somewhat involved, it represents a good improvement to the possible analytic solution for Ising models with more than one neighbour interaction. Furthermore this general approach might be useful in other problems for computing eigenvalues for matrices with higher dimension.

References

Marchi E and Millan L 1977 *Proc. First Int. Conf. on Mathematical Modelling* vol 2, 447–58
Thompson C J 1972 *Mathematical Statistical Mechanics* (New York: McMillan)